Logic Programming
A Category - Theoretic Framework

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I Categories

A *category* is a directed graph whose nodes are called *objects* and whose edges are called *arrows*, equipped with a partial operation on arrows called *composition* satisfying the following conditions

- For each object A there is an arrow $A \xrightarrow{id_A} A$ called the *identity arrow* on A.

- For each *compatible* pair of arrows $A \xrightarrow{f} B, B \xrightarrow{g} C$ (meaning $src(g) = target(f)$) there is an arrow $A \xrightarrow{fg} C$ called the *composition* of f and g

- composition (when defined) is associative: $(fg)h = f(gh)$

- For all objects A, B, C, and arrows $C \xrightarrow{f} A, A \xrightarrow{g} C$

$$f \, id_A = f \qquad id_A \, g = g$$

Examples

- Set : the category with sets as objects and functions as arrows.
- Ab: objects: Abelian Groups, arrows: group homomorphisms
- Grp: objects: Groups, arrows: group homomorphisms
- Top: objects: Topological Spaces, arrows: continuous maps
- $\mathbb{A} \times \mathbb{B}$: objects: pairs $(A, B) \in |\mathbb{A}| \times |\mathbb{B}|$, arrows: pairs of arrows from $\mathbb{A}, \mathbb{B}$.
- **1**, the category with one object and one arrow.
- **2**, the category $\cdot \longrightarrow \cdot$ with two objects, their identity arrows and one arrow between them.

- Rel: objects: sets, arrows: binary relations $A \xrightarrow{R} B$.
- $\mathbb{A}^o$: The *opposite* category of $\mathbb{A}$. objects: the objects of $\mathbb{A}$, arrows: $B \xrightarrow{f^o} A$ for each $A \xrightarrow{f} B$. So for

$$A \xrightarrow{f} B \xrightarrow{g} C \text{ we have } C \xrightarrow{g^o} B \xrightarrow{f^o} A$$
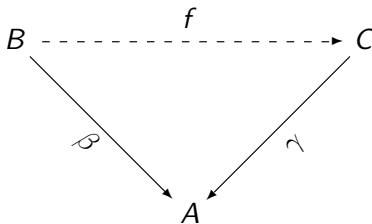
  with $g^o f^o = (fg)^o$ and $id^o = id$.
- Graph: objects: Graphs, arrows: graph homomorphisms.

An important example: the slice category

Let $\mathbb{C}$ be a category, and A an object of $\mathbb{C}$. Then the slice of $\mathbb{C}$ by A (or $\mathbb{C}$ over A), written $\mathbb{C}/A$, is the category with

- objects: arrows of $\mathbb{C}$ whose target is A.
- arrows: from $B \xrightarrow{\beta} A$ to $C \xrightarrow{\gamma} A$ are arrows f in $\mathbb{C}$ from B to C making the following diagram commute

Reversing the arrows gives rise to the *coslice* $A/\mathbb{C}$.

Note that there is a natural functor $\mathbb{C} \longrightarrow \mathbb{C}/A$. (send B to $B \times A \xrightarrow{\ r\ } A$) Action on arrows?.

Some categorical notions

- A terminal object in $\mathbb{C}$ is an object (called **1**) such that for every other object A there is a unique arrow $A \xrightarrow{\ !\ } \mathbf{1}$.
- A coterminal (or initial) object **0** in $\mathbb{B}$ satisfies the *dual* property: for any object A there is a unique arrow $\mathbf{0} \xrightarrow{\ ?\ } A$

What are the initial and terminal objects in Set, Rel, $\mathbb{C}/A$?

Monic and Epic Arrows

- An arrow $A \xrightarrow{m} B$ in a category is *monic* if for every pair of arrows

$$\bullet \underset{y}{\overset{x}{\rightrightarrows}} A \xrightarrow{m} B$$

if $xm = ym$ then $x = y$.

- An arrow $A \xrightarrow{e} B$ in a category is *epic* if for every pair of arrows

$$A \xrightarrow{e} B \underset{y}{\overset{x}{\rightrightarrows}} \bullet$$

if $mx = my$ then $x = y$.

In Set epis are precisely the surjections, and monos the injections. Not necessarily so in other concrete categories. e.g. in the category Mon of Monoids. Consider containment $\mathbb{N} \xrightarrow{\subseteq} \mathbb{Z}$. It's an epi!.

Products and Coproducts

A product of two objects A and B in a category is an object $A \times B$ together with a diagram $A \xleftarrow{\;l\;} A \times B \xrightarrow{\;r\;} B$ (or just the triple $(A \times B, l, r)$) satisfying the following condition. For every object C and pair of arrows $A \xleftarrow{\;f\;} C \xrightarrow{\;g\;} B$ there is a unique arrow $\langle f, g \rangle : C \longrightarrow A \times B$ making the following diagram commute
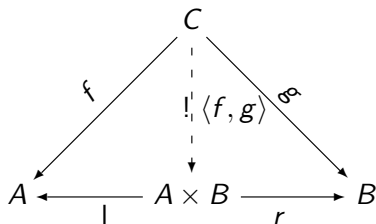
A coproduct of two objects A, B in category $\mathbb{A}$ is an object together with a diagram $A \xrightarrow{\iota_l} A + B \xleftarrow{\iota_r} B$ satisfying the following condition For every object C and pair of arrows $A \xrightarrow{f} C \xleftarrow{g} B$ there is a unique arrow $[f, g] : C \longleftarrow A + B$ making the following diagram commute
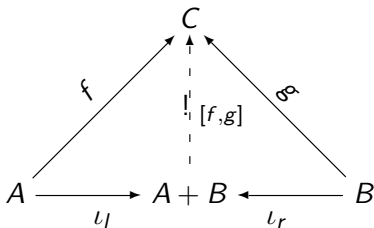
Let $\mathbb{A}$ be a category with products and let A, B be objects of $\mathbb{A}$ and $\mathbb{C}$ be the category whose objects are diagrams of the form $A \xleftarrow{f} C \xrightarrow{g} B$. with an arrow $(A \xleftarrow{f} C \xrightarrow{g} B) \xrightarrow{\varphi} (A \xleftarrow{f'} C' \xrightarrow{g'} B)$ defined to be an arrow $C \xrightarrow{\varphi} C'$ of $\mathbb{C}$ making the induced diagram commute. What is the terminal object in this category?

Examples

- In Set, $\mathbf{0}$ is $\{\}$, $\mathbf{1}$ is $\{*\}$, any one element set.
- In Set $A \times B$ is the Cartesian Product, $A + B$ the disjoint union.
- in Rel, $\mathbf{0}$ is the same as in Set, which is also $\mathbf{1}_{Rel}$. $\star$what is the product? (it's not the cartesian product), coproduct?
- Top, the category of topological spaces and continuous maps has the sum (with the sum topology = the finest topology making the injections continuous) as a coproduct. The product is just the set-theoretic product together with the so-called product topology **not** the box topology.

Functors

Let $\mathbb{A}, \mathbb{B}$ be categories. A *functor* $F : \mathbb{A} \longrightarrow B$ is given by a pair of functions $F : |\mathbb{A}| \longrightarrow |\mathbb{B}|$ and $F : arr(\mathbb{A}) \longrightarrow arr(\mathbb{B})$ satisfying:

$$
\begin{aligned}
F(C \xrightarrow{f} D) &= F(C) \xrightarrow{F(f)} F(D) \\
F(fg) &= F(f)F(g) \\
F(id_A) &= id_{F(A)}
\end{aligned}
$$

A *contravariant* functor from $\mathbb{A}$ to $\mathbb{B}$ is just a functor
$F : \mathbb{A}^o \longrightarrow \mathbb{B}$. We take this to mean, for objects C, D in $\mathbb{A}$

$$
\begin{aligned}
F(C \xrightarrow{\ f\ } D) &= F(D) \xrightarrow{\ F(f)\ } F(C) \\
F(fg) &= F(g)F(f) \\
F(id_A) &= id_{F(A)}
\end{aligned}
$$

Cat, the category of categories has categories as objects and functors as arrows.

An arrow between two objects $A \xrightarrow{f} B$ is an *isomorphism* if it has both a left and right inverse, i.e. if there is an arrow $B \xrightarrow{g} A$ such that $fg = id_A$ and $gf = id_B$.

A functor can also be an isomorphism (it is an arrow in the category of categories). e.g. just as in Set we have

$$\mathbb{A} \times \mathbf{1} \simeq \mathbb{A} \qquad (\mathbb{A} \times \mathbb{B}) \times \mathbb{C} \simeq \mathbb{A} \times (\mathbb{B} \times \mathbb{C}) \qquad \mathbb{A} \times \mathbb{B} \simeq \mathbb{B} \times \mathbb{A}$$

Functors formalize some important correspondences in mathematics. For example the homotopy group functior

$$\text{Top} \xrightarrow{\ \pi\ } \text{Grp}$$

Sending topological spaces to homotopy - equivalent loop classes. Continuous maps between topological spaces are sent to group homomorphisms.
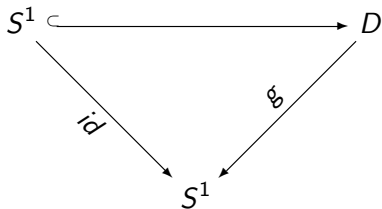
This functor can be used to give a simple proof of Brouwer's fixed point theorem

Suppose f is a continuous map $D \longrightarrow D$ that does not fix any point x.
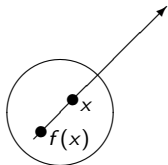
Let $D \xrightarrow{\ g\ } \partial D$ send each point x to the point on the boundary picked out by the vector shown above.

The function g is continuous and maps each point on the boundary to itself.

$$\begin{array}{ccc} S^1 & \longrightarrow & D \\ & & \\ {\scriptstyle id} \searrow & & \swarrow {\scriptstyle g} \\ & S^1 & \end{array}$$

The induced diagram cannot commute. Contradiction.

$$\mathbb{Z} = \pi(S^1) \longrightarrow \pi(D) = \{0\}$$

$$id \qquad \pi(g)$$

$$\mathbb{Z} = \pi(S^1)$$

- Any monoid can be viewed as a one object category. Functors between them are monoid homomorphisms.
- Similarly groups are one-object categories with all arrows isomorphisms.
- Posets are categories with at most one arrow between objects (and the identity arrow on each).

Comma categories

The slice is a special case of the comma category construction: Let $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ be categories with functors

$$F : \mathbb{A} \longrightarrow \mathbb{C} \text{ and } G : \mathbb{B} \longrightarrow \mathbb{C}.$$

An object in the *comma category* $(F \downarrow G)$ is an arrow from $F(A)$ to $G(B)$ for some objects A, B. An arrow between two objects u, v is a pair of arrows f, g (of the appropriate type) making the following digram commute.

$$
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(A') \\
\downarrow u & & \downarrow v \\
G(B) & \xrightarrow[G(g)]{} & G(B')
\end{array}
$$

The slice $\mathbb{C}/A$ is a special case: $(\mathbf{1}_{\mathbb{C}} \downarrow \lambda x.A)$.

Natural Transformations

There is a natural way to define a mapping from one functor to another (with the same type). Let $F, G : \mathbb{A} \longrightarrow \mathbb{B}$. A *natural transformation* α from F to G is a family of arrows $\{\alpha_X : F(X) \longrightarrow G(X) | X \in |\mathbb{A}|\}$ in $\mathbb{B}$, one for each object X of $\mathbb{A}$ satisfying the following "naturality condition": For each arrow $X \xrightarrow{m} Y$ in $\mathbb{A}$ the following diagram commutes.

$$
\begin{array}{ccc}
F(X) & \xrightarrow{\ \alpha_X\ } & G(X) \\
{\scriptstyle F(m)}\Big\downarrow & & \Big\downarrow{\scriptstyle G(m)} \\
F(Y) & \xrightarrow{\ \alpha_Y\ } & G(Y)
\end{array}
$$

Functor categories

Given two categories $\mathbb{A}, \mathbb{B}$ we can the define the *functor category* $\mathbb{B}^{\mathbb{A}}$ with

objects: Functors from $\mathbb{A}$ to $\mathbb{B}$

arrows: Natural transformations from one functor to another.

Many interesting mathematical structures arise as functor categories. Let $\mathbb{M}$ be a monoid defined as a one-object category. Then $\text{Set}^{\mathbb{M}}$ is the category of $\mathbb{M}$-sets, or *semigroup actions*. That is to say, each object (i.e. functor) F picks out a set $F(\bullet)$ and a closed family of functions on this set. We can define, for $x \in F(\bullet)$, $m \in \mathbb{M}$

$$xm := F(m)(x)$$

Then it is easy to check that functoriality of F guarantees the monoid action axioms:

$$x(m_1 m_2) = (xm_1)m_2 \qquad x\,id_{\bullet} = x$$

Natural transformations η between $\mathbb{M}$-sets are $\mathbb{M}$-homomorphisms: $\eta(xm) = \eta(x)m$.

A group acting on a set is similarly formalized as a functor category.

The functor category

$$\mathrm{Set}^{\bullet_1 \;\substack{\longrightarrow \\ \longrightarrow}\; \bullet_2}$$

can be seen as the category of *graphs*. Each object is essentially a pair of sets (edges and nodes) with a pair of maps between them, namely the *src* and *target* maps.

A natural transformation α between two objects F and G in this category is a pair of maps, $\alpha_{edge}, \alpha_{node}$ satisfying

$$
\begin{array}{ccc}
Edge_F & \xrightarrow{\;\alpha_{edge}\;} & Edge_G \\[2pt]
\Big\downarrow src_F & & \Big\downarrow src_G \\[2pt]
Node_F & \xrightarrow{\;\alpha_{node}\;} & Node_G
\end{array}
$$

and the same commutativity for the target map. α is a graph homomorphism.

A nice $\star$exercise:

For any categories $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$,

$$\mathbb{A}^{\mathbf{1}} \simeq \mathbb{A} \qquad \mathbb{C}^{\mathbb{A} \times \mathbb{B}} \simeq (\mathbb{C}^{\mathbb{B}})^{\mathbb{A}} \qquad (\mathbb{A} \times \mathbb{B})^{\mathbb{C}} \simeq \mathbb{A}^{\mathbb{C}} \times \mathbb{B}^{\mathbb{C}}$$

For example, for the second problem let's see how to define,

- for each functor $F : \mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{C}$ a functor $F^* : \mathbb{A} \longrightarrow \mathbb{C}^{\mathbb{B}}$, and

- for each nat $t : F \longrightarrow G$, a nat $t^* : F^* \longrightarrow G^*$.

Let $F^*(A)(B) = F(A, B)$ and $F^*(A)(B_1 \xrightarrow{\beta} B_2) = F(id_A, \beta)$.
Now for any arrow $A_1 \xrightarrow{\alpha} A_2$ in $\mathbb{A}$ we define
$F^*(\alpha) : F^*(A_1) \longrightarrow F^*(A_2)$ to be the nat
$F^*(\alpha)_B : F^*(A_1)(B) \longrightarrow F^*(A_2)(B) = F(\alpha, id_B)$.
Finally we need to show how $()^*$ acts on natural transformations
$t : F \longrightarrow G$ in $\mathbb{C}^{\mathbb{A} \times \mathbb{B}}$. Let $(t_A^*)_B = t_{(A,B)}$. Finally show $(\ )^*$ has a
2-sided inverse. The rest is left as an exercise.

Composing functors and natural transformations

Given

$$\mathbb{D} \xrightarrow{\ L\ } \mathbb{A} \underset{H}{\overset{F}{\underset{\longrightarrow}{\rightrightarrows}}} \mathbb{B} \xrightarrow{\ K\ } \mathbb{C}$$

and natural transformations $t : F \longrightarrow G$ and $u : G \longrightarrow H$, we define the compositions

- $(Lt)_D = t_{L(D)} : LF(D) \longrightarrow LG(D)$
- $(tK)_A = K(t_A) : FK(A) \longrightarrow GK(A)$

These compositions satisfy the following laws (exercise)

$$(tu)K = K(t)K(u) \qquad L(tu) = (Lt)(Lu)$$

If the category $\mathbb{A}$ (the collection of arrows) is a *set* then $\mathbb{A}$ is said to be *small*.

$\mathbb{A}$ is said to be *locally small* if for every pair of objects A, B the collection of arrows from A to B, denoted $\mathbb{A}(A, B)$ or $Hom_{\mathbb{A}}(A, B)$, is a *set*.

A category $\mathbb{B}$ is said to be a *subcategory* of $\mathbb{A}$ if $|\mathbb{B}|$ is contained in $|\mathbb{A}|$ and for every pair of objects B_1, B_2 of $\mathbb{B}$

$$\mathbb{B}(B_1, B_2) \text{ is contained in } \mathbb{A}(B_1, B_2)$$

$\mathbb{B}$ is said to be a *full* subcategory of $\mathbb{A}$ if for every pair of objects B_1, B_2 of $\mathbb{B}$
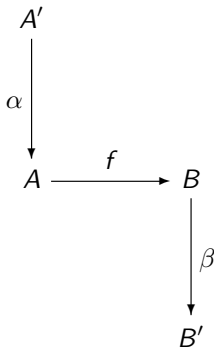
$$\mathbb{B}(B_1, B_2) \text{ is equal to } \mathbb{A}(B_1, B_2)$$

Ex.: Ab is a full subcategory of Grp.

Hom-functors

If $\mathbb{C}$ is a locally small category, then we have several functors to Set induced by Hom-sets:

- $Hom_{\mathbb{C}} : \mathbb{C}^o \times \mathbb{C} \longrightarrow$ Set given by $(A, B) \mapsto \mathbb{C}(A, B)$ on objects and for $A' \xrightarrow{\alpha} A$ and $B \xrightarrow{\beta} B'$ in $\mathbb{C}$, $\mathbb{C}(\alpha, \beta)$ is the function $\mathbb{C}(A, B) \longrightarrow \mathbb{C}(A', B')$ given by $f \mapsto \alpha f \beta$.

$$
\begin{array}{ccc}
A' & & \\
\downarrow{\scriptstyle\alpha} & & \\
A & \xrightarrow{f} & B \\
& & \downarrow{\scriptstyle\beta} \\
& & B'
\end{array}
$$

Assume $\mathbb{C}$ is locally small.

- For each object B of $\mathbb{C}$ we have a *contravariant* functor $Hom_{\mathbb{C}}(\_, B) : \mathbb{C} \longrightarrow$ Set given by

$$
\begin{aligned}
A &\mapsto Hom_{\mathbb{C}}(A, B) \\
A' \xrightarrow{\alpha} A &\mapsto Hom_{\mathbb{C}}(\alpha, B) : Hom_{\mathbb{C}}(A, B) \xrightarrow{\lambda f. \alpha f} Hom_{\mathbb{C}}(A', B)
\end{aligned}
$$

- For each object A of $\mathbb{C}$ we have a *covariant* (i.e. normal) functor $Hom_{\mathbb{C}}(A, \_) : \mathbb{C} \longrightarrow$ Set given by

$$
\begin{array}{rcl}
B & \mapsto & Hom_{\mathbb{C}}(A, B) \\
B \xrightarrow{\ \beta\ } B' & \mapsto & Hom_{\mathbb{C}}(A, \beta) : Hom_{\mathbb{C}}(A, B) \xrightarrow{\ \lambda f. f\beta\ } Hom_{\mathbb{C}}(A, B')
\end{array}
$$

Therefore the correspondence

$$B \mapsto Hom_{\mathbb{C}}(\text{-}, B)$$

gives rise to a *covariant* functor

$$\mathbf{y} : \mathbb{C} \longrightarrow Set^{\mathbb{C}^o}$$

known as the *Yoneda embedding*.

The action of **y** on arrows $B \xrightarrow{\beta} B'$ is $\beta \mapsto Hom_{\mathbb{C}}(\_, \beta)$, the latter being the natural transformation

$$\mathbf{y}(\beta) : Hom_{\mathbb{C}}(\_, B) \longrightarrow Hom_{\mathbb{C}}(\_, B')$$

whose action $\mathbf{y}(\beta)_A : Hom_{\mathbb{C}}(A, B) \longrightarrow Hom_{\mathbb{C}}(A, B')$ is post-composition with β:

$$\mathbf{y}(\beta)_A(f) = f\beta$$

Let $Nat(X, Y)$ denote the collection of natural transformations from X to Y.

If $\mathbb{C}$ is locally small, and $F : \mathbb{C}^o \longrightarrow$ Set then for each $B \in |\mathbb{C}|$

$$Nat(Hom_{\mathbb{C}}(\_, B), F) \simeq F(B)$$

Proof: (sketch) Refer to the diagram below, assuming
$f : A \longrightarrow B \in \mathbb{C}$
Define $\hat{()} : Nat(Hom_{\mathbb{C}}(\_, B), F) \longrightarrow F(B)$ by $\hat{\lambda} = \lambda_B(id_B)$

$$
\begin{array}{ccc}
id_B \in Hom(B, B) & \xrightarrow{\lambda_B} & F(B) \\
\Big\downarrow {\scriptstyle Hom(f, B)} & & \Big\downarrow {\scriptstyle F(f)} \\
Hom(A, B) & \xrightarrow{\lambda_A} & F(A)
\end{array}
$$

Now define $\dot{()} : F(B) \longrightarrow Nat(Hom_{\mathbb{C}}(\_, B), F)$ by letting $\dot{b}$ be the natural transformation $\dot{b}_A(f) = F(f)(b)$.

It is straightforward to show that the two maps are inverses. ($\star$ Exercise) Interestingly they are both *natural in B and F*.

Taking F to be $Hom(\_, A)$ for some A in $|\mathbb{A}|$ we have the following special case:

$$Nat(Hom(\_, B), Hom(\_, A)) \simeq Hom(B, A)$$

i.e. every nat corresponds to an arrow.

Definition

A functor F is *faithful* if it is injective on arrows, i.e. if the induced mapping: $Hom(A, B) \longrightarrow Hom(F(A), F(B))$ is injective. It is *full* if this mapping is surjective. F is a *full embedding* if full, faithful and injective on objects.

Theorem

The Yoneda functor is a full embedding

⋆Exercise. See e.g. *Lambek-Scott*

One of the most important ideas in Category Theory is that of adjoint functors. We will first look at how they are defined for posets (or even pre-ordered sets), that is to say for categories where there is at most one arrow $p \leq q$ between objects. Recall that functors between posets are monotone maps.

Def. Let $\mathbb{A}$ and $\mathbb{B}$ be posets, $F : \mathbb{A} \longrightarrow \mathbb{B}$ and $G : \mathbb{B} \longrightarrow \mathbb{A}$ functors. F is *left-adjoint* to G (written $F \dashv G$) if for all x, y

$$F(x) \leq y \quad \text{iff} \quad x \leq G(y)$$

Such a pair is also called a *Galois correspondence* in the pre-order case given.

Adjoint Functors

⋆Notice that such a Galois correspondence gives rise to a *closure operation* FG (i.e. $G \circ F$) satisfying

$$x \leq FG(x) \qquad FGFG(x) \leq FG(x) \qquad x \leq y \Rightarrow FG(x) \leq FG(y)$$

Also, for $a \in \mathbb{A}$ and $b \in \mathbb{B}$

$$FGFG(a) \simeq FG(a) \text{ and } GFGF(b) \simeq GF(b)$$

i.e. FG and GF create fixed points.

Thus, a Galois correspondence $F \dashv G$ determines an *equivalence*
between the pre-ordered set $\mathbb{A}_0$ of fixed points of FG and the
pre-order $\mathbb{B}_0$ of fixed points of GF i.e. between the "closed"
elements of $\mathbb{A}$ and the "open" elements of $\mathbb{B}$.

$$
\begin{array}{ccc}
\mathbb{A} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathbb{B} \\
\Big\uparrow {\scriptstyle \subseteq} & & \Big\uparrow {\scriptstyle \subseteq} \\
\mathbb{A}_0 & \underset{\simeq}{\rightleftarrows} & \mathbb{B}_0
\end{array}
$$

We will revisit this in a categorical setting soon.

Let $\mathbb{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{B}$ be a pair of functors. F, G are an *adjoint pair* $\equiv F$ is a left adjoint to $G \equiv G$ is a right adjoint to F if there is a *bijection*

$$\mathbb{B}(FA, B) \simeq \mathbb{A}(A, GB)$$

natural in A and B for every object A in $\mathbb{A}$ and B in $\mathbb{B}$. Equivalently there is a natural isomorphism:

$$\mathbb{B}(F(\_), \_) \simeq \mathbb{A}(\_, G(\_)) : \mathbb{A}^o \times \mathbb{B} \longrightarrow \text{Set}.$$

Adjunction is signalled with the following notation:

$$F \dashv G$$

For each set B let Let $B \times \_ : \text{Set} \longrightarrow \text{Set}$ and
$(\_)^B : \text{Set} \longrightarrow \text{Set}$ be defined by

$$B \times \_(A) = B \times A \text{ and } (\_)^B(C) = C^B$$

The "famous maps" *curry* and *uncurry* give inverse bijections
$C^{B \times A} \simeq (C^B)^A$ i.e. between $\text{Set}(B \times A, C)$ and $\text{Set}(A, C^B)$

Let Prop be the poset category whose objects are propositions with arrows $F \longrightarrow G$ given by entailment. (This category will be formalized later in the course).

Let $\Delta : \text{Prop} \longrightarrow \text{Prop} \times \text{Prop}$ (the so-called *diagonal functor*) be given by $\Delta(A) = (A, A)$, with the obvious action on arrows.

Define $\vee, \wedge$ to be functors $\text{Prop} \times \text{Prop} \longrightarrow \text{Prop}$ with the actions: $\vee(A, B) = A \vee B$ etc. and straightforward corresponding actions on arrows. Then

$$\Delta \dashv \wedge \text{ and } \vee \dashv \Delta$$

Because we (easily) have

$$\frac{(A, A) \longrightarrow (B, C)}{A \longrightarrow B \wedge C}$$

and

$$\frac{B \vee C \longrightarrow A}{(B, C) \longrightarrow (A, A)}$$

Let $B \wedge \_$ and $B \supset \_$ be the obvious functors from Prop to Prop. They are adjoints (just like the proof of $B \times \_ \dashv ()^B$).

Let U be the "forgetful functor" (also called the "Underlying" functor) from Set to Grp, the functor that maps each group to itself as a *set* forgetting the algebraic structure.

Let S be a set. An object G in Grp is called a free object on S in its category if there is an arrow $S \xrightarrow{g} U(G)$ and for every group H and every arrow (in Set) $S \xrightarrow{h} U(H)$ a *unique* group homomorphism $G \xrightarrow{\varphi} H$ making the following diagram commute:


This is called the *universal mapping property* of free groups.

For every set S a free group $F(S)$ on S exists (with the map from S to $U(FS)$ inclusion). Just take the set S' consisting of all so-called words (strings) on $S \cup \{s^{-1} : s \in S\}$ with multiplication given by concatenation but with adjacent "inverses" canceling. F is really a functor: Set $\longrightarrow$ Grp. It is immediate to check that there is an adjunction
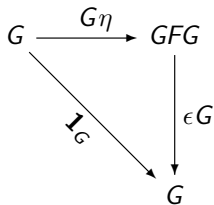
$$F \dashv U$$

An equivalent formulation of adjunction is the following. Given functors $\mathbb{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{B}$ we say that an adjunction (or adjoint situation) is a four-tuple $\langle F, G, \eta, \epsilon \rangle$ where $\eta : \mathbf{1}_{\mathbb{A}} : \longrightarrow FG$ and $\epsilon : GF \longrightarrow \mathbf{1}_{\mathbb{B}}$ are natural transformations called the *unit* and *counit* of the adjunction respectively, making the following diagrams commute.
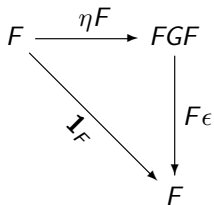
i.e. $(\eta F)(F\epsilon) = 1_F$, and

$(G\eta)(\epsilon G) = 1_G$

Theorem

The two formulations are equivalent.

The unit and co-unit are easily obtained:

$$\frac{F(A) \xrightarrow{id} F(A)}{A \xrightarrow{\eta_A} FG(A)} \qquad \frac{G(B) \xrightarrow{id} G(B)}{GF(B) \xrightarrow{\epsilon_B} B}.$$

Their naturality follows easily from the naturality of the bijection $\mathbb{B}(FA, B) \simeq \mathbb{A}(A, GB)$. The verification of the diagrams is left as an ⋆exercise.

A solution to the *universal mapping problem* for a functor
$U : \mathbb{B} \longrightarrow \mathbb{A}$ is given by the following data:
For each object A of $\mathbb{A}$: an object $F(A)$ of $\mathbb{B}$ and an arrow
$\eta_A : A \longrightarrow FU(A)$ such that for each object B of $\mathbb{B}$ and each
arrow $f : A \longrightarrow U(B)$ in $\mathbb{A}$ there is a unique arrow
$f^* : F(A) \longrightarrow B$ in $\mathbb{B}$ such that $\eta(A)U(f^*) = f$.

$$
\begin{array}{ccc}
A \xrightarrow{\ \eta_A\ } U(F(A)) & & F(A) \\
\\
\big\downarrow{\scriptstyle U(f^*)} & & \big\downarrow{\scriptstyle !\ f^*} \\
\\
U(B) & & B
\end{array}
$$

with arrow $f : A \longrightarrow U(B)$

If $\mathbb{B}$ is a full subcategory of $\mathbb{A}$ and U is the inclusion, then we say that $\mathbb{B}$ is a reflective subcategory of $\mathbb{A}$. (i.e. when inclusion has a left adjoint). Remark: [Lambek-Scott] We can think of $\eta_A : A \longrightarrow F(A)$ as giving the best approximation in $\mathbb{B}$ of the object A of $\mathbb{A}$.

(famous examples: Abelian groups and Groups. Sheaves and presheaves).

Theorem

There is a bijection between adjoint situations $\langle F, U, \eta, \epsilon \rangle$ and solutions $(F, \eta, *)$ of the universal mapping problem.

[see *Lambek-Scott, MacLane*]

Definition

An equivalence between categories $\mathbb{A}$ and $\mathbb{B}$ is a pair of functors

$$\mathbb{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{B}$$

where FU and UF are naturally isomorphic to the identity

$$FU \simeq \mathbf{1}_{\mathbb{A}} \qquad UF \simeq \mathbf{1}_{\mathbb{B}}$$

An equivalence gives rise to an adjunction in which the unit and counit are isomorphisms (called an *adjoint equivalence*).

Theorem (Freyd, Lambek-Scott)

An adjunction $\langle F, U, \eta, \epsilon \rangle$ induces an equivalence between the following full subcategories $\mathbb{A}_{\nvdash}$ of $\mathbb{A}$ and $\mathbb{B}_0$ of $\mathbb{B}$

$$
\begin{aligned}
\mathbb{A}_0 &\equiv \; Fix\eta \equiv \; \{A \in |\mathbb{A}| : \eta_A \text{ is an iso} \} \\
\mathbb{B}_0 &\equiv \; Fix\eta \equiv \; \{B \in |\mathbb{B}| : \epsilon_B \text{ is an iso} \}
\end{aligned}
$$

ηU is an iso iff $F\epsilon$ is.

Examples abound in mathematics: $\mathbb{A} = $ Rings, $\mathbb{B} = $ Top^o and Stone duality. The correspondence between Sheaves and Local Homeomorphisms.

Terminal objects and products are examples of limits in a category. So are pullbacks and equalizers which we now define.

Definition

Let $A \underset{n}{\overset{m}{\rightrightarrows}} B$ be a pair of arrows in a category $\mathbb{A}$. An *equalizer* of this pair of arrows (i.e. of this diagram) is an object E together with an arrow $E \xrightarrow{\ e\ } A$ such that

$$E \xrightarrow{\ e\ } A \underset{n}{\overset{m}{\rightrightarrows}}$$

commutes and such that for any other arrow $C \xrightarrow{\ f\ } A$ satisfying $fm = fn$ there is a unique arrow φ_E making the following diagram commute

$$E \xrightarrow{\ e\ } A \underset{n}{\overset{m}{\rightrightarrows}} B$$

with φ_E and f from C.
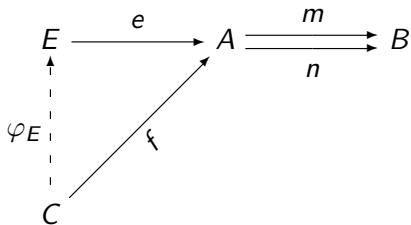
Coequalizers: dualize the diagram.

Pullbacks (produits fibrés)

Given

$$B$$

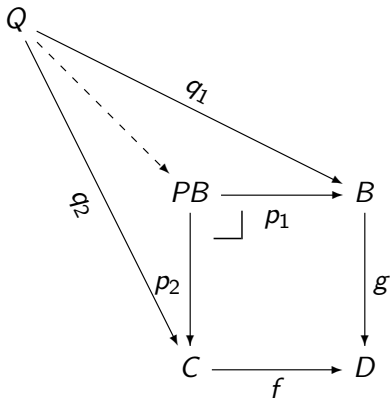$$\downarrow g$$

$$C \xrightarrow{\quad f \quad} D$$

a pullback PB is a diagram $C \xleftarrow{\;p_2\;} PB \xrightarrow{\;p_1\;} A$ such that

$$
\begin{array}{ccc}
PB & \xrightarrow{\;p_1\;} & B \\
\downarrow{\scriptstyle p_2} & \lrcorner & \downarrow{\scriptstyle g} \\
C & \xrightarrow{\;f\;} & D
\end{array}
$$

and given any other such diagram

$$Q \xrightarrow{\quad q_1 \quad} B$$

$$q_2 \downarrow \qquad \qquad \downarrow g$$

$$C \xrightarrow{\quad f \quad} D$$
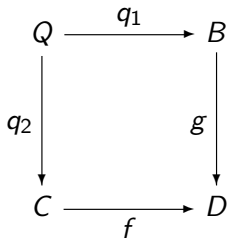
There is a unique $Q \dashrightarrow PB$ making the resulting diagrams created commute.

q_2 is often called *the pullback of g along f* (and similarly with *f* and q_1).
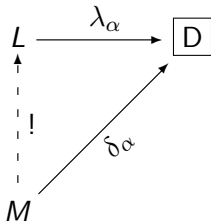
The pullback of a monic is monic. The same with isos.

Idea: given a diagram **D**, a limit is an object L and a family of arrows making all created diagrams commute (called a *cone* over **D**)

$$L \xrightarrow{\lambda_\alpha} \boxed{D}$$

Such that given any other

$$M \xrightarrow{\delta_\alpha} \boxed{D}$$

There is a unique $M \dashrightarrow L$ making all resulting diagrams commute

I.e. a *limit* is a terminal cone.

So what's a diagram in a category $\mathbb{A}$? We can formalize it as a *functor* Δ from a category $\mathbb{I}$ (sometimes called the index category) into $\mathbb{A}$. Then we can define a *cone* in $\mathbb{A}$ as a pair (A, η) where $A \in |\mathbb{A}|$ and η is a natural transformation

$$\eta : \lambda x.A \longrightarrow \Delta$$

from the constant A-valued functor: $\mathbb{I} \xrightarrow{\lambda x.A} \mathbb{A}$ to Δ. We then say that the *functor* Δ has a limit (L, λ) if it is a terminal cone, i.e. if for any other (Q, ν) there is a unique arrow $Q \longrightarrow L$ making all created triangles commute.

Thus, informally,

- a terminal object is a limit of $\{\bullet\}$.
- an equalizer is a limit of $\bullet \rightrightarrows \bullet$
- a product is a limit of $\{\bullet \qquad \bullet\}$
- a pullback is a limit of $\{\bullet \longleftarrow \bullet \longrightarrow \bullet\}$

Dualize the definitions to obtain: initial object, coequalizer, coproduct, pushout (somme amalgamé), colimit.

Some exercises:

- ▶ Limits are terminal objects in which category?
- ▶ Limits are unique up to isomorphism (any two limits of a given diagram are isomorphic).
- ▶ Set has all small limits and colimits (i.e. it is *complete*).
- ▶ Left adjoints preserve colimits, right adjoints preserve limits.

Let $\mathbb{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{B}$ be an adjoint pair and let $\tau : \lambda x.a \longrightarrow \Delta$ be a

limiting cone (we write this as: $a \overset{\cdot}{\longrightarrow} \Delta$ following MacLane). Each arrow in the limiting cone is of the form $\tau(i) : a \longrightarrow \Delta(i)$. Apply the functor G to this cone. We will show we get a limiting cone: $\tau G : G(a) \overset{\cdot}{\longrightarrow} \Delta G$
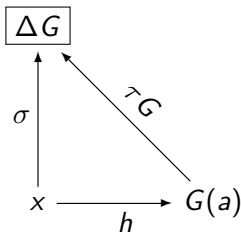
Let $\sigma : x \longrightarrow \Delta G$ be a cone over ΔG. Each arrow $\sigma_i : x \longrightarrow G(\Delta(i))$ gives rise (by the adjunction) to an arrow $\sigma_i^* : F(x) \longrightarrow \Delta(i)$ i.e. a cone $\sigma^* : F(x) \longrightarrow \Delta$ over Δ. But τ is a limiting cone. So there exists a unique arrow $h : F(x) \longrightarrow a$ making all induced diagrams commute (ie. $h\tau = \sigma^*$).
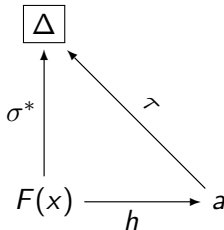
By adjunction we get a unique arrow $h_* : x \longrightarrow G(a)$. One must check that $(h\tau)_* = h_*(\tau G) = \sigma$.

So h is a unique arrow making all diagrams commute. Hence $\tau G : G(a) \xrightarrow{\quad} (\Delta G)$ is a limiting cone, as we wanted to show.

Some other important results about limits

- $Hom(A, \_) : \mathbb{C} \longrightarrow$ Set preserves all existing limits.
- Set$^{\mathbb{C}^o}$ is complete.
- A category with a terminal object and pullbacks is complete.
- A category with equalizers and all small products is complete.

(see Borceux)

Monads

We saw that in a Galois correspondence $F \dashv G$ between $\mathbb{A}$ and $\mathbb{B}$ we can conclude that for every A in $\mathbb{A}$

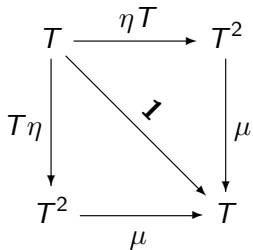$$A \leq FG(A) \text{ and } FGFG(A) \leq FG(A)$$

We have similar results for adjoints. A straightforward argument from the definition yields natural transformations $\mathbf{1}_{\mathbb{A}} \longrightarrow FG$ (namely the unit) and $FGFG \longrightarrow FG$. These properties are of independent interest.

A Functor $T : \mathbb{A} \longrightarrow \mathbb{B}$ is a monad if it is equipped with natural transformations
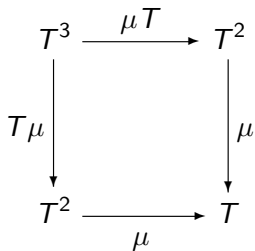
- $\eta : \mathbf{1}_{\mathbb{A}} \longrightarrow T$, and
- $\mu : TT \longrightarrow T$

making the following diagrams commute.

and

Theorem
If $F \dashv G$ then $\langle FG, \eta, F\epsilon G \rangle$ is a monad.

- From $GF \xrightarrow{\epsilon} id$ we get
- $FGF \xrightarrow{\epsilon_F} F$, i.e. $FGF \xrightarrow{F\epsilon} F$. Now apply G
- $FGFG \xrightarrow{F\epsilon G} FG$.

Recall: The adjunction (F, G, η, ϵ) satisfies $(\eta F)(F\epsilon) = 1_F$ and $(G\eta)(\epsilon G) = 1_G$

We must now check the commutativity of

$$
\begin{array}{ccc}
FG & \xrightarrow{\ \eta FG\ } & FGFG \\
{\scriptstyle FG\eta} \downarrow & {\scriptstyle 1_{FG}} \searrow & \downarrow {\scriptstyle F\epsilon G} \\
FGFG & \xrightarrow{\ F\epsilon G\ } & FG
\end{array}
$$

i.e.

- $(\eta FG)(F\epsilon G) = 1_{FG}$, which, by functoriality of G is $(\eta F)(F\epsilon)G$ which is $\mathbf{1}_F G$, i.e. 1_{FG}, and

- $(FG\eta)(F\epsilon G) = 1_{FG}$ which is also immediate:
  $F(G\eta)(\epsilon G) = F1_G = 1_{FG}$.

Examples

- The closure operator in a topological space: Let $T(S)$ for any subset S of X be the intersection of all closed sets containing S. $T : \langle P(X), \subseteq \rangle \longrightarrow \langle P(X), \subseteq \rangle$ is a monad: $S \subseteq T(S)$ and $T(T(S)) \subseteq T(S)$

- Let $\mathcal{M}$ be a monoid with unit 1 and underlying set M. For each set S define $T(A) = M \times A$ and $\eta_A : A \longrightarrow M \times A$ via $\eta_A(a) = (a, 1)$ and $\mu(A) : M \times (M \times A) \longrightarrow M \times A$ via $\mu(m_1, (m_2, a)) \longrightarrow (m_1 m_2, a)$.

  Let $T(A_1 \xrightarrow{\alpha} A_2)$ be the function $(m, a) \mapsto (m, \alpha(a))$.

  $\langle (\rangle T, \eta, \mu)$ defines a monad. The monad laws here stipulate $m1 = m = 1m$ and $(m_1 m_2) m_3 = m_1 (m_2 m_3)$ (hence the names *unity* and *associative* laws).

More examples

Let P be the covariant power set functor $\mathrm{Set} \longrightarrow \mathrm{Set}$ mapping sets A to $P(A)$ and maps $A \xrightarrow{f} B$ to the function $P(f) : P(A) \xrightarrow{Im_f} P(B)$. Define the natural transformations $\eta : 1 \longrightarrow P$ and $\mu : PP \longrightarrow P$ by: $\eta_A(x) = \{x\}$ and $\mu S = \bigcup S$ sending each subset of A to its image under f. (This map is sometimes called $\exists_f$).

Does every monad arise as a composition of adjoint functors?

Adjoint resolutions

The answer is yes, given a monad (T, η, μ) on $\mathbb{A}$ there is a *category* of so-called *adjoint resolutions* $(\mathbb{B}, F, G, \epsilon)$ of a monad. These are given by an adjoint pair of functors F, G from $\mathbb{A}$ to $\mathbb{B}$ with unit η and counit ϵ satisfying $F\epsilon G = \mu$. Arrows in this category are morphisms $\Phi : (\mathbb{B}, F, G, \epsilon) \longrightarrow (\mathbb{B}', F', G', \epsilon')$ that satisfy:
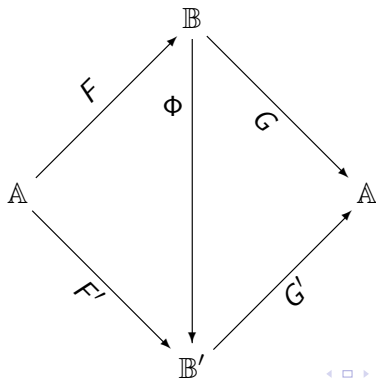
The category of adjoint resolutions of T has

- A initial object $\mathbb{A}_T$ called the *Kleisli* category of T (the category of free T-algebras)
- A final object $\mathbb{A}^T$ called the Eilenberg-Moore category of T (the category of T-algebras).

Given an adjoint resolution $(\mathbb{B}, F, G, \epsilon)$ for T, the unique arrows $K^T : \mathbb{B} \longrightarrow \mathbb{A}^T$ and $K_T : \mathbb{A}_T \longrightarrow \mathbb{B}$ are called the *comparison functors*. The case where the comparison functors give rise to an equivalence of categories is of special interest. (See MacLane or Lambek-Scott for details).

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