Logic Programming

A Category - Theoretic Framework

James Lipton (Wesleyan)

I Categories

A *category* is a directed graph whose nodes are called *objects* and whose edges are called *arrows*, equipped with a partial operation on arrows called *composition* satisfying the following conditions

- For each object A there is an arrow $A \xrightarrow{id_A} A$ called the *identity arrow* on A.
- For each *compatible* pair of arrows $A \xrightarrow{f} B, B \xrightarrow{g} C$ (meaning $src(g) = target(f)$) there is an arrow $A \xrightarrow{fg} C$ called the *composition* of f and g
- composition (when defined) is associative: $(fg)h = f(gh)$
- For all objects A, B, C, and arrows $C \xrightarrow{f} A, A \xrightarrow{g} C$

$$f\, id_A = f \qquad id_A\, g = g$$

Examples

- Set : the category with sets as objects and functions as arrows.
- Ab: objects: Abelian Groups, arrows: group homomorphisms
- Grp: objects: Groups, arrows: group homomorphisms
- Top: objects: Topological Spaces, arrows: continuous maps
- $\mathbb{A} \times \mathbb{B}$: objects: pairs $(A, B) \in |\mathbb{A}| \times |\mathbb{B}|$, arrows: pairs of arrows from $\mathbb{A}, \mathbb{B}$.
- $\mathbf{1}$, the category with one object and one arrow.
- $\mathbf{2}$, the category $\cdot \longrightarrow \cdot$ with two objects, their identity arrows and one arrow between them.

More examples

- Rel: objects: sets, arrows: binary relations $A \xrightarrow{R} B$.
- $\mathbb{A}^{\mathrm{o}}$: The *opposite* category of $\mathbb{A}$. objects: the objects of $\mathbb{A}$, arrows: $B \xrightarrow{f^o} A$ for each $A \xrightarrow{f} B$. So for

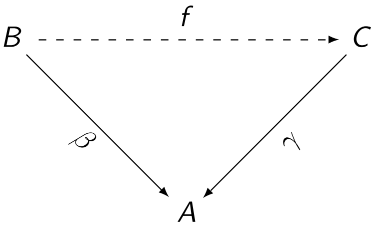
  $$A \xrightarrow{f} B \xrightarrow{g} C \text{ we have } C \xrightarrow{g^o} B \xrightarrow{f^o} A$$

  with $g^o f^o = (fg)^o$ and $id^o = id$.
- Graph: objects: Graphs, arrows: graph homomorphisms.

An important example: the slice category

Let $\mathbb{C}$ be a category, and A an object of $\mathbb{C}$. Then the slice of $\mathbb{C}$ by A (or $\mathbb{C}$ over A), written $\mathbb{C}/A$, is the category with

- objects: arrows of $\mathbb{C}$ whose target is A.
- arrows: from $B \xrightarrow{\beta} A$ to $C \xrightarrow{\gamma} A$ are arrows f in $\mathbb{C}$ from B to C making the following diagram commute



Reversing the arrows gives rise to the *coslice* $A/\mathbb{C}$.

Note that there is a natural functor $\mathbb{C} \longrightarrow \mathbb{C}/A$. (send B to $B \times A \xrightarrow{r} A$) Action on arrows?.

Some categorical notions

- A terminal object in $\mathbb{C}$ is an object (called $\mathbf{1}$) such that for every other object A there is a unique arrow $A \xrightarrow{!} \mathbf{1}$.
- A coterminal (or initial) object $\mathbf{0}$ in $\mathbb{B}$ satisfies the *dual* property: for any object A there is a unique arrow $\mathbf{0} \xrightarrow{?} A$

What are the initial and terminal objects in Set, Rel, $\mathbb{C}/A$?

Monic and Epic Arrows

- An arrow $A \xrightarrow{m} B$ in a category is *monic* if for every pair of arrows

$$\bullet \overset{x}{\underset{y}{\rightrightarrows}} A \xrightarrow{m} B$$

  if $xm = ym$ then $x = y$.

- An arrow $A \xrightarrow{e} B$ in a category is *epic* if for every pair of arrows

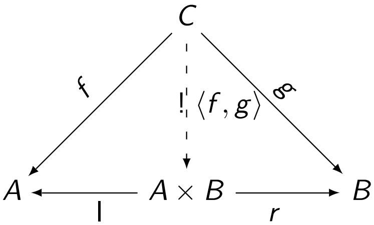
$$A \xrightarrow{e} B \overset{x}{\underset{y}{\rightrightarrows}} \bullet$$

  if $mx = my$ then $x = y$.

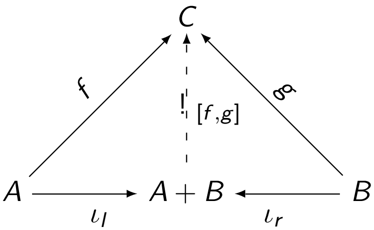
In Set epis are precisely the surjections, and monos the injections. Not necessarily so in other concrete categories. e.g. in the category Mon of Monoids. Consider containment $\mathbb{N} \xrightarrow{\subseteq} \mathbb{Z}$. It's an epi!.

Products and Coproducts

A product of two objects A and B in a category is an object $A \times B$ together with a diagram $A \xleftarrow{\mathsf{l}} A \times B \xrightarrow{r} B$ (or just the triple $(A \times B, \mathsf{l}, r)$) satisfying the following condition. For every object C and pair of arrows $A \xleftarrow{f} C \xrightarrow{g} B$ there is a unique arrow $\langle f, g \rangle : C \longrightarrow A \times B$ making the following diagram commute



A coproduct of two objects A,B in category $\mathbb{A}$ is an object together with a diagram $A \xrightarrow{\iota_l} A+B \xleftarrow{\iota_r} B$ satisfying the following condition For every object C and pair of arrows $A \xrightarrow{f} C \xleftarrow{g} B$ there is a unique arrow $[f,g] : C \longleftarrow A+B$ making the following diagram commute



Let $\mathbb{A}$ be a category with products and let A, B be objects of $\mathbb{A}$ and $\mathbb{C}$ be the category whose objects are diagrams of the form $A \xleftarrow{f} C \xrightarrow{g} B$. with an arrow $(A \xleftarrow{f} C \xrightarrow{g} B) \xrightarrow{\varphi} (A \xleftarrow{f'} C' \xrightarrow{g'} B)$ defined to be an arrow $C \xrightarrow{\varphi} C'$ of $\mathbb{C}$ making the induced diagram commute. What is the terminal object in this category?

Examples

- In Set, $\mathbf{0}$ is $\{\}$, $\mathbf{1}$ is $\{*\}$, any one element set.
- In Set $A \times B$ is the Cartesian Product, $A + B$ the disjoint union.
- in Rel, $\mathbf{0}$ is the same as in Set, which is also $\mathbf{1}_{Rel}$. $\star$what is the product? (it's not the cartesian product), coproduct?
- Top, the category of topological spaces and continuous maps has the sum (with the sum topology = the finest topology making the injections continuous) as a coproduct. The product is just the set-theoretic product together with the so-called product topology **not** the box topology.

Functors

Let $\mathbb{A}, \mathbb{B}$ be categories. A *functor* $F : \mathbb{A} \longrightarrow B$ is given by a pair of functions $F : |\mathbb{A}| \longrightarrow |\mathbb{B}|$ and $F : arr(\mathbb{A}) \longrightarrow arr(\mathbb{B})$ satisfying:

$$
\begin{aligned}
F(C \xrightarrow{f} D) &= F(C) \xrightarrow{F(f)} F(D) \\
F(fg) &= F(f)F(g) \\
F(id_A) &= id_{F(A)}
\end{aligned}
$$

Functors (cont)

A *contravariant* functor from $\mathbb{A}$ to $\mathbb{B}$ is just a functor $F : \mathbb{A}^{\mathrm{o}} \longrightarrow \mathbb{B}$. We take this to mean, for objects C, D in $\mathbb{A}$

$$\begin{aligned} F(C \xrightarrow{f} D) &= F(D) \xrightarrow{F(f)} F(C) \\ F(fg) &= F(g)F(f) \\ F(id_A) &= id_{F(A)} \end{aligned}$$

Functors (cont)

Cat, the category of categories has categories as objects and functors as arrows.

isos

An arrow between two objects $A \xrightarrow{f} B$ is an *isomorphism* if it has both a left and right inverse, i.e. if there is an arrow $B \xrightarrow{g} A$ such that $fg = id_A$ and $gf = id_B$.
A functor can also be an isomorphism (it is an arrow in the category of categories). e.g. just as in Set we have

$$\mathbb{A} \times \mathbf{1} \simeq \mathbb{A} \qquad (\mathbb{A} \times \mathbb{B}) \times \mathbb{C} \simeq \mathbb{A} \times (\mathbb{B} \times \mathbb{C}) \qquad \mathbb{A} \times \mathbb{B} \simeq \mathbb{B} \times \mathbb{A}$$

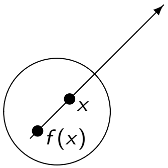
A “famous” functor: The fundamental group

Functors formalize some important correspondences in mathematics. For example the homotopy group functior

$$\text{Top} \xrightarrow{\pi} \text{Grp}$$

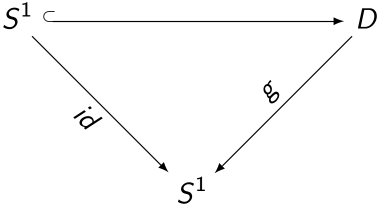
Sending topological spaces to homotopy - equivalent loop classes.
Continuous maps between topological spaces are sent to group homomorphisms.
This functor can be used to give a simple proof of Brouwer’s fixed point theorem

Suppose f is a continuous map $D \longrightarrow D$ that does not fix any point x.

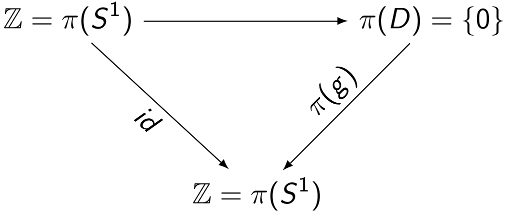


Let $D \xrightarrow{g} \partial D$ send each point x to the point on the boundary picked out by the vector shown above.

The function g is continuous and maps each point on the boundary to itself.



The induced diagram cannot commute. Contradiction.



Some algebraic examples

- Any monoid can be viewed as a one object category. Functors between them are monoid homomorphisms.
- Similarly groups are one-object categories with all arrows isomorphisms.
- Posets are categories with at most one arrow between objects (and the identity arrow on each).

Comma categories

The slice is a special case of the comma category construction: Let $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ be categories with functors

$$F : \mathbb{A} \longrightarrow \mathbb{C} \text{ and } G : \mathbb{B} \longrightarrow \mathbb{C}.$$

An object in the *comma category* $(F \downarrow G)$ is an arrow from $F(A)$ to $G(B)$ for some objects A, B. An arrow between two objects u, v is a pair of arrows f, g (of the appropriate type) making the following digram commute.

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ \Big\downarrow u & & \Big\downarrow v \\ G(B) & \xrightarrow[G(g)]{} & G(B') \end{array}$$

The slice $\mathbb{C}/A$ is a special case: $(\mathbf{1}_{\mathbb{C}} \downarrow \lambda x.A)$.

Natural Transformations

There is a natural way to define a mapping from one functor to another (with the same type). Let $F, G : \mathbb{A} \longrightarrow \mathbb{B}$. A *natural transformation* α from F to G is a family of arrows $\{\alpha_X : F(X) \longrightarrow G(X) | X \in |\mathbb{A}|\}$ in $\mathbb{B}$, one for each object X of $\mathbb{A}$ satisfying the following "naturality condition": For each arrow $X \xrightarrow{m} Y$ in $\mathbb{A}$ the following diagram commutes.

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ {\scriptstyle F(m)}\downarrow & & \downarrow{\scriptstyle G(m)} \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array}$$

Functor categories

Given two categories $\mathbb{A}, \mathbb{B}$ we can the define the *functor category* $\mathbb{B}^{\mathbb{A}}$ with

objects: Functors from $\mathbb{A}$ to $\mathbb{B}$

arrows: Natural transformations from one functor to another.

Functor Categories (cont)

Many interesting mathematical structures arise as functor categories. Let $\mathbb{M}$ be a monoid defined as a one-object category. Then $\mathsf{Set}^{\mathbb{M}}$ is the category of $\mathbb{M}$-sets, or *semigroup actions*. That is to say, each object (i.e. functor) F picks out a set $F(\bullet)$ and a closed family of functions on this set. We can define, for $x \in F(\bullet)$, $m \in \mathbb{M}$

$$xm := F(m)(x)$$

Then it is easy to check that functoriality of F guarantees the monoid action axioms:

$$x(m_1 m_2) = (xm_1)m_2 \qquad xid_\bullet = x$$

Natural transformations η between $\mathbb{M}$-sets are $\mathbb{M}$-homomorphisms: $\eta(xm) = \eta(x)m$.
A group acting on a set is similarly formalized as a functor category.

Functor Categories (cont)

The functor category

$$\mathsf{Set}^{\bullet_1 \rightrightarrows \bullet_2}$$

can be seen as the category of *graphs*. Each object is essentially a pair of sets (edges and nodes) with a pair of maps between them, namely the *src* and *target* maps.

A natural transformation α between two objects F and G in this category is a pair of maps, $\alpha_{edge}, \alpha_{node}$ satisfying

$$\begin{array}{ccc} Edge_F & \xrightarrow{\alpha_{edge}} & Edge_G \\ {\scriptstyle src_F}\downarrow & & \downarrow{\scriptstyle src_G} \\ Node_F & \xrightarrow{\alpha_{node}} & Node_G \end{array}$$

and the same commutativity for the target map. α is a graph homomorphism.

More on functor cats and natural transformations

A nice $\star$exercise:
For any categories $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$,

$$\mathbb{A}^{\mathbf{1}} \simeq \mathbb{A} \qquad \mathbb{C}^{\mathbb{A}\times\mathbb{B}} \simeq (\mathbb{C}^{\mathbb{B}})^{\mathbb{A}} \qquad (\mathbb{A}\times\mathbb{B})^{\mathbb{C}} \simeq \mathbb{A}^{\mathbb{C}} \times \mathbb{B}^{\mathbb{C}}$$

For example, for the second problem let's see how to define,

- for each functor $F : \mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{C}$ a functor $F^* : \mathbb{A} \longrightarrow \mathbb{C}^{\mathbb{B}}$, and
- for each nat $t : F \longrightarrow G$, a nat $t^* : F^* \longrightarrow G^*$.

Let $F^*(A)(B) = F(A, B)$ and $F^*(A)(B_1 \xrightarrow{\beta} B_2) = F(id_A, \beta)$.
Now for any arrow $A_1 \xrightarrow{\alpha} A_2$ in $\mathbb{A}$ we define
$F^*(\alpha) : F^*(A_1) \longrightarrow F^*(A_2)$ to be the nat
$F^*(\alpha)_B : F^*(A_1)(B) \longrightarrow F^*(A_2)(B) = F(\alpha, id_B)$.
Finally we need to show how $()^*$ acts on natural transformations
$t : F \longrightarrow G$ in $\mathbb{C}^{\mathbb{A}\times\mathbb{B}}$. Let $(t^*_A)_B = t_{(A,B)}$. Finally show ( )* has a 2-sided inverse. The rest is left as an exercise.

Composing functors and natural transformations

Given

$$\mathbb{D} \xrightarrow{\ L\ } \mathbb{A} \;\begin{matrix} \xrightarrow{\ F\ } \\ \xrightarrow{\ G\ } \\ \xrightarrow[\ H\ ]{} \end{matrix}\; \mathbb{B} \xrightarrow{\ K\ } \mathbb{C}$$

and natural transformations $t : F \longrightarrow G$ and $u : G \longrightarrow H$, we define the compositions

- $(Lt)_D = t_{L(D)} : LF(D) \longrightarrow LG(D)$
- $(tK)_A = K(t_A) : FK(A) \longrightarrow GK(A)$

These compositions satisfy the following laws (exercise)

$$(tu)K = K(t)K(u) \qquad L(tu) = (Lt)(Lu)$$

Locally small and small categories

If the category $\mathbb{A}$ (the collection of arrows) is a *set* then $\mathbb{A}$ is said to be *small*.

$\mathbb{A}$ is said to be *locally small* if for every pair of objects A, B the collection of arrows from A to B, denoted $\mathbb{A}(A, B)$ or $Hom_{\mathbb{A}}(A, B)$, is a *set*.

Subcategories

A category $\mathbb{B}$ is said to be a *subcategory* of $\mathbb{A}$ if $|\mathbb{B}|$ is contained in $|\mathbb{A}|$ and for every pair of objects B_1, B_2 of $\mathbb{B}$

$$\mathbb{B}(B_1, B_2) \text{ is contained in } \mathbb{A}(B_1, B_2)$$

$\mathbb{B}$ is said to be a *full* subcategory of $\mathbb{A}$ if for every pair of objects B_1, B_2 of $\mathbb{B}$

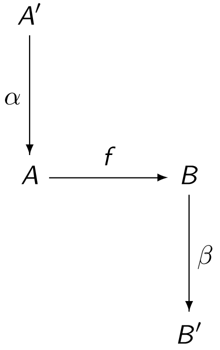
$$\mathbb{B}(B_1, B_2) \text{ is equal to } \mathbb{A}(B_1, B_2)$$

Ex.: Ab is a full subcategory of Grp.

Hom-functors

If $\mathbb{C}$ is a locally small category, then we have several functors to Set induced by Hom-sets:

- $Hom_{\mathbb{C}} : \mathbb{C}^o \times \mathbb{C} \longrightarrow$ Set given by $(A, B) \mapsto \mathbb{C}(A, B)$ on objects and for $A' \xrightarrow{\alpha} A$ and $B \xrightarrow{\beta} B'$ in $\mathbb{C}$, $\mathbb{C}(\alpha, \beta)$ is the function $\mathbb{C}(A, B) \longrightarrow \mathbb{C}(A', B')$ given by $f \mapsto \alpha f \beta$.



Assume $\mathbb{C}$ is locally small.

- For each object B of $\mathbb{C}$ we have a *contravariant* functor $Hom_{\mathbb{C}}(\_, B) : \mathbb{C} \longrightarrow$ Set given by

$$\begin{array}{rcl} A & \mapsto & Hom_{\mathbb{C}}(A, B) \\ A' \xrightarrow{\alpha} A & \mapsto & Hom_{\mathbb{C}}(\alpha, B) : Hom_{\mathbb{C}}(A, B) \xrightarrow{\lambda f.\alpha f} Hom_{\mathbb{C}}(A', B) \end{array}$$

- For each object A of $\mathbb{C}$ we have a *covariant* (i.e. normal) functor $Hom_{\mathbb{C}}(A, \_) : \mathbb{C} \longrightarrow \mathsf{Set}$ given by

$$\begin{array}{rcl} B & \mapsto & Hom_{\mathbb{C}}(A, B) \\ B \xrightarrow{\beta} B' & \mapsto & Hom_{\mathbb{C}}(A, \beta) : Hom_{\mathbb{C}}(A, B) \xrightarrow{\lambda f . f \beta} Hom_{\mathbb{C}}(A, B') \end{array}$$

Yoneda

Therefore the correspondence

$$B \mapsto Hom_{\mathbb{C}}(\_, B)$$

gives rise to a *covariant* functor

$$\mathbf{y} : \mathbb{C} \longrightarrow \mathsf{Set}^{\mathbb{C}^o}$$

known as the *Yoneda embedding*.

The action of $\mathbf{y}$ on arrows $B \xrightarrow{\beta} B'$ is $\beta \mapsto Hom_{\mathbb{C}}(\_,\beta)$, the latter being the natural transformation

$$\mathbf{y}(\beta) : Hom_{\mathbb{C}}(\_,B) \longrightarrow Hom_{\mathbb{C}}(\_,B')$$

whose action $\mathbf{y}(\beta)_A : Hom_{\mathbb{C}}(A, B) \longrightarrow Hom_{\mathbb{C}}(A, B')$ is post-composition with β:

$$\mathbf{y}(\beta)_A(f) = f\beta$$

The Yoneda Lemma

Let $Nat(X, Y)$ denote the collection of natural transformations from X to Y.
If $\mathbb{C}$ is locally small, and $F : \mathbb{C}^o \longrightarrow \text{Set}$ then for each $B \in |\mathbb{C}|$

$$Nat(Hom_{\mathbb{C}}(\_, B), F) \simeq F(B)$$

Proof: (sketch) Refer to the diagram below, assuming $f : A \longrightarrow B \in \mathbb{C}$

Define $\hat{()} : Nat(Hom_{\mathbb{C}}(\_, B), F) \longrightarrow F(B)$ by $\hat{\lambda} = \lambda_B(id_B)$

$$\begin{array}{ccc} id_B \in Hom(B, B) & \xrightarrow{\lambda_B} & F(B) \\ \Big\downarrow {\scriptstyle Hom(f, B)} & & \Big\downarrow {\scriptstyle F(f)} \\ Hom(A, B) & \xrightarrow{\lambda_A} & F(A) \end{array}$$

Now define $\dot{()} : F(B) \longrightarrow Nat(Hom_{\mathbb{C}}(\_, B), F)$ by letting $\dot{b}$ be the natural transformation $\dot{b}_A(f) = F(f)(b)$.
It is straightforward to show that the two maps are inverses. ($\star$ Exercise) Interestingly they are both *natural in B and F*.

Yoneda: corollary

Taking F to be $Hom(\_, A)$ for some A in $|\mathbb{A}|$ we have the following special case:

$$Nat(Hom(\_, B), Hom(\_, A)) \simeq Hom(B, A)$$

i.e. every nat corresponds to an arrow.

Definition

A functor F is *faithful* if it is injective on arrows, i.e. if the induced mapping: $Hom(A, B) \longrightarrow Hom(F(A), F(B))$ is injective. It is *full* if this mapping is surjective. F is a full embedding if full, faithful and injective on objects.

Theorem

The Yoneda functor is a full embedding

$\star$Exercise. See e.g. *Lambek-Scott*

Adjoint Functors

In posets

One of the most important ideas in Category Theory is that of adjoint functors. We will first look at how they are defined for posets (or even pre-ordered sets), that is to say for categories where there is at most one arrow $p \leq q$ between objects. Recall that functors between posets are monotone maps.

Def. Let $\mathbb{A}$ and $\mathbb{B}$ be posets, $F : \mathbb{A} \longrightarrow \mathbb{B}$ and $G : \mathbb{B} \longrightarrow \mathbb{A}$ functors. F is *left-adjoint* to G (written $F \dashv G$) if for all x, y

$$F(x) \leq y \quad \text{iff } x \leq G(y)$$

Such a pair is also called a *Galois correspondence* in the pre-order case given.

Adjoint Functors

$\star$Notice that such a Galois correspondence gives rise to a *closure* operation FG (i.e. $G \circ F$) satisfying

$$x \leq FG(x) \qquad FGFG(x) \leq FG(x) \qquad x \leq y \Rightarrow FG(x) \leq FG(y)$$

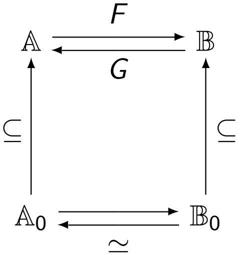
Also, for $a \in \mathbb{A}$ and $b \in \mathbb{B}$

$$FGFG(a) \simeq FG(a) \text{ and } GFGF(b) \simeq GF(b)$$

i.e. FG and GF *create fixed points*.

Equivalences

Thus, a Galois correspondence $F \dashv G$ determines an *equivalence* between the pre-ordered set $\mathbb{A}_0$ of fixed points of FG and the pre-order $\mathbb{B}_0$ of fixed points of GF i.e. between the "closed" elements of $\mathbb{A}$ and the "open" elements of $\mathbb{B}$.



We will revisit this in a categorical setting soon.

Adjunctions

Let $\mathbb{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{B}$ be a pair of functors. F, G are an *adjoint pair* $\equiv$ F is a left adjoint to G $\equiv$ G is a right adjoint to F if there is a *bijection*

$$\mathbb{B}(FA, B) \simeq \mathbb{A}(A, GB)$$

natural in A and B for every object A in $\mathbb{A}$ and B in $\mathbb{B}$. Equivalently there is a natural isomorphism:

$$\mathbb{B}(F(\_), \_) \simeq \mathbb{A}(\_, G(\_)) : \mathbb{A}^o \times \mathbb{B} \longrightarrow \text{Set}.$$

Adjunction is signalled with the following notation:

$$F \dashv G$$

$$\frac{A \longrightarrow G(B)}{F(A) \longrightarrow B}$$

Examples

For each set B let Let $B \times \_ : \mathrm{Set} \longrightarrow \mathrm{Set}$ and $(\_)^B : \mathrm{Set} \longrightarrow \mathrm{Set}$ be defined by

$$B \times \_(A) = B \times A \text{ and } (\_)^B(C) = C^B$$

The "famous maps" *curry* and *uncurry* give inverse bijections $C^{B\times A} \simeq (C^B)^A$ i.e. between $\mathrm{Set}(B \times A, C)$ and $\mathrm{Set}(A, C^B)$

Let Prop be the poset category whose objects are propositions with arrows $F \longrightarrow G$ given by entailment. (This category will be formalized later in the course).
Let $\Delta : \text{Prop} \longrightarrow \text{Prop} \times \text{Prop}$ (the so-called *diagonal functor*) be given by $\Delta(A) = (A, A)$, with the obvious action on arrows.
Define $\vee, \wedge$ to be functors $\text{Prop} \times \text{Prop} \longrightarrow \text{Prop}$ with the actions: $\vee(A, B) = A \vee B$ etc. and straightforward corresponding actions on arrows. Then

$$\Delta \dashv \wedge \text{ and } \vee \dashv \Delta$$

Because we (easily) have

$$\frac{(A, A) \longrightarrow (B, C)}{A \longrightarrow B \wedge C}$$

and

$$\frac{B \vee C \longrightarrow A}{(B, C) \longrightarrow (A, A)}$$

The adjunction of *wedge* and $\supset$

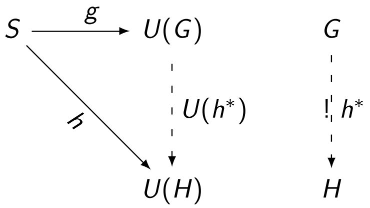
Let $B \wedge \_$ and $B \supset \_$ be the obvious functors from Prop to Prop.
They are adjoints (just like the proof of $B \times \_ \dashv ()^B$).

Free objects

...on sets

Let U be the "forgetful functor" (also called the "Underlying" functor) from Set to Grp, the functor that maps each group to itself as a *set* forgetting the algebraic structure.

Let S be a set. An object G in Grp is called a free object on S in its category if there is an arrow $S \xrightarrow{g} U(G)$ and for every group H and every arrow (in Set) $S \xrightarrow{h} U(H)$ a *unique* group homomorphism $G \xrightarrow{h^*} H$ making the following diagram commute:



This is called the *universal mapping property* of free groups.

For every set S a free group $F(S)$ on S exists (with the map from S to $U(FS)$ inclusion). Just take the set S' consisting of all so-called words (strings) on $S \cup \{s^{-1} : s \in S\}$ with multiplication given by concatenation but with adjacent “inverses” canceling. F is really a functor: Set $\longrightarrow$ Grp. It is immediate to check that there is an adjunction

$$F \dashv U$$

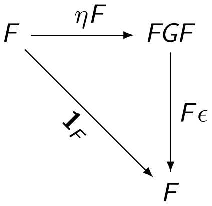
Examples of Free objects

Every Vector space is free on (any) basis.

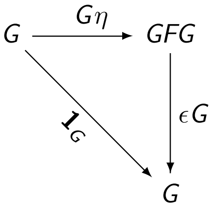
There is a free category on any (small) graph.

Adjoints: an alternative formulation

An equivalent formulation of adjunction is the following. Given functors $\mathbb{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{B}$ we say that an adjunction (or adjoint situation) is a four-tuple $\langle F, G, \eta, \epsilon \rangle$ where $\eta : \mathbf{1}_{\mathbb{A}} : \longrightarrow FG$ and $\epsilon : GF \longrightarrow \mathbf{1}_{\mathbb{B}}$ are natural transformations called the *unit* and *counit* of the adjunction respectively, making the following diagrams commute.



i.e. $(\eta F)(F\epsilon) = 1_F$, and



$(G\eta)(\epsilon G) = 1_G$

Theorem

The two formulations are equivalent.

The unit and co-unit are easily obtained:

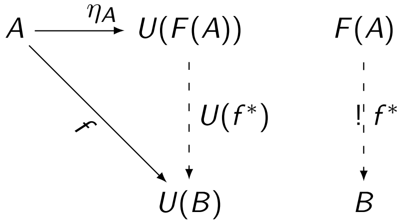
$$\frac{F(A) \xrightarrow{id} F(A)}{A \xrightarrow{\eta_A} FG(A)} \qquad \frac{G(B) \xrightarrow{id} G(B)}{GF(B) \xrightarrow{\epsilon_B} B}.$$

Their naturality follows easily from the naturality of the bijection $\mathbb{B}(FA, B) \simeq \mathbb{A}(A, GB)$. The verification of the diagrams is left as an $\star$exercise.

A third

Essentially the free object formulation

A solution to the *universal mapping problem* for a functor $U : \mathbb{B} \longrightarrow \mathbb{A}$ is given by the following data:
For each object A of $\mathbb{A}$: an object $F(A)$ of $\mathbb{B}$ and an arrow $\eta_A : A \longrightarrow FU(A)$ such that for each object B of $\mathbb{B}$ and each arrow $f : A \longrightarrow U(B)$ in $\mathbb{A}$ there is a unique arrow $f^* : F(A) \longrightarrow B$ in $\mathbb{B}$ such that $\eta(A)U(f^*) = f$.



If $\mathbb{B}$ is a full subcategory of $\mathbb{A}$ and U is the inclusion, then we say that $\mathbb{B}$ is a reflective subcategory of $\mathbb{A}$. (i.e. when inclusion has a left adjoint). Remark: [Lambek-Scott] We can think of $\eta_A : A \longrightarrow F(A)$ as giving the best approximation in $\mathbb{B}$ of the object A of $\mathbb{A}$.
(famous examples: Abelian groups and Groups. Sheaves and presheaves).

Theorem

There is a bijection between adjoint situations $\langle F, U, \eta, \epsilon \rangle$ *and solutions* $(F, \eta, *)$ *of the universal mapping problem.*

[see *Lambek-Scott, MacLane*]

Equivalence of Categories

Definition

An equivalence between categories $\mathbb{A}$ and $\mathbb{B}$ is a pair of functors

$$\mathbb{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{B}$$

where FU and UF are naturally isomorphic to the identity

$$FU \simeq \mathbf{1}_{\mathbb{A}} \qquad UF \simeq \mathbf{1}_{\mathbb{B}}$$

An equivalence gives rise to an adjunction in which the unit and counit are isomorphisms (called an *adjoint equivalence*).

Theorem (Freyd, Lambek-Scott)

An adjunction $\langle F, U, \eta, \epsilon \rangle$ *induces an equivalence between the following full subcategories* $\mathbb{A}_0$ *of* $\mathbb{A}$ *and* $\mathbb{B}_0$ *of* $\mathbb{B}$

$$\begin{aligned} \mathbb{A}_0 \equiv \quad & Fix\eta \equiv \quad \{A \in |\mathbb{A}| : \eta_A \text{ is an iso } \} \\ \mathbb{B}_0 \equiv \quad & Fix\eta \equiv \quad \{B \in |\mathbb{B}| : \epsilon_B \text{ is an iso } \} \end{aligned}$$

$U\eta$ *is an iso iff* $F\epsilon$ *is.*

Examples abound in mathematics: $\mathbb{A} = \text{Rings}$, $\mathbb{B} = \text{Top}^o$ and Stone duality. The correspondence between Sheaves and Local Homeomorphisms.

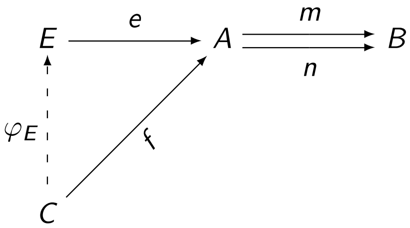
Terminal objects and products are examples of limits in a category. So are pullbacks and equalizers which we now define.

Definition

Let $A \underset{n}{\overset{m}{\rightrightarrows}} B$ be a pair of arrows in a category $\mathbb{A}$. An *equalizer* of this pair of arrows (i.e. of this diagram) is an object E together with an arrow $E \xrightarrow{e} A$ such that

$$E \xrightarrow{e} A \underset{n}{\overset{m}{\rightrightarrows}}$$

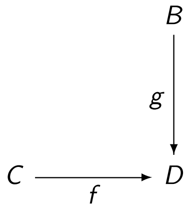
commutes and such that for any other arrow $C \xrightarrow{f} A$ satisfying $fm = fn$ there is a unique arrow φ_E making the following diagram commute



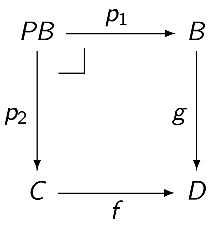
Coequalizers: dualize the diagram.

Pullbacks (produits fibrés)

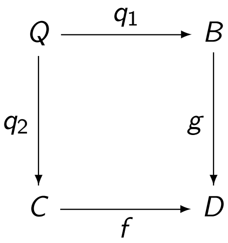
Given



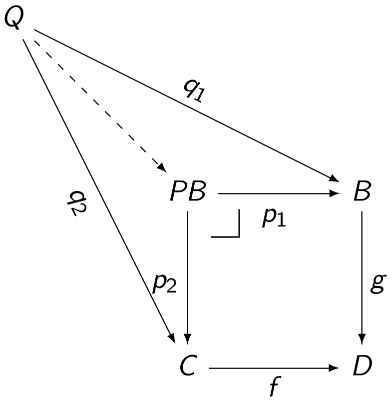
a pullback PB is a diagram $C \xleftarrow{p_2} PB \xrightarrow{p_1} A$ such that



and given any other such diagram



There is a unique $Q \dashrightarrow PB$ making the resulting diagrams created commute.



p_2 is often called *the pullback of g along f* (and similarly with f and p_1).

$\star$ Exercise

The pullback of a monic is monic. The same with isos.

Limits

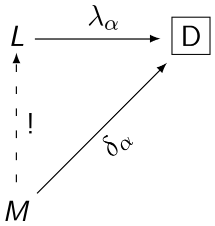
Idea: given a diagram **D**, a limit is an object L and a family of arrows into **D** making all created diagrams commute (called a *cone* over **D**)

$$L \xrightarrow{\lambda_\alpha} \boxed{\text{D}}$$

Such that given any other

$$M \xrightarrow{\delta_\alpha} \boxed{\text{D}}$$

There is a unique $M \dashrightarrow L$ making all resulting diagrams commute



I.e. a *limit* is a terminal cone.

Formalizing the definition a bit...

So what's a diagram in a category $\mathbb{A}$? We can formalize it as a *functor* Δ from a category $\mathbb{I}$ (sometimes called the index category) into $\mathbb{A}$. Then we can define a *cone* in $\mathbb{A}$ as a pair (A, η) where $A \in |\mathbb{A}|$ and η is a natural transformation

$$\eta : \lambda x.A \longrightarrow \Delta$$

from the constant A-valued functor:$\mathbb{I} \xrightarrow{\lambda x.A} \mathbb{A}$ to Δ. We then say that the *functor* Δ has a limit (L, λ) if it is a terminal cone, i.e. if for any other (Q, ν) there is a unique arrow $Q \longrightarrow L$ making all created triangles commute.

Thus, informally,

- a terminal object is a limit of $\{\_\}$.
- an equalizer is a limit of $\bullet \rightrightarrows \bullet$
- a product is a limit of $\{\bullet \quad \bullet\}$
- a pullback is a limit of $\{\bullet \longleftarrow \bullet \longrightarrow \bullet\}$

Dualize the definitions to obtain: initial object, coequalizer, coproduct, pushout (somme amalgamé), colimit.

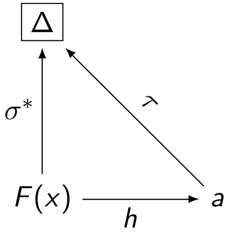
Some exercises:

- Limits are terminal objects in which category?
- Limits are unique up to isomorphism (any two limits of a given diagram are isomorphic).
- Set has all small limits and colimits (i.e. it is *complete*).
- Left adjoints preserve colimits, right adjoints preserve limits.

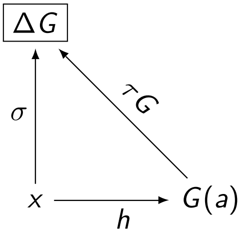
Right adjoints preserve limits

Let $\mathbb{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{B}$ be an adjoint pair and let $\tau : \lambda x.a \longrightarrow \Delta$ be a limiting cone (we write this as: $a \overset{\cdot}{\longrightarrow} \Delta$ following MacLane). Each arrow in the limiting cone is of the form $\tau(i) : a \longrightarrow \Delta(i)$. Apply the functor G to this cone. We will show we get a limiting cone: $\tau G : G(a) \overset{\cdot}{\longrightarrow} \Delta G$

Let $\sigma : x \stackrel{\cdot}{\longrightarrow} \Delta G$ be a cone over ΔG. Each arrow $\sigma_i : x \longrightarrow G(\Delta(i))$ gives rise (by the adjunction) to an arrow $\sigma_i^* : F(x) \longrightarrow \Delta(i)$ i.e. a cone $\sigma^* : F(x) \stackrel{\cdot}{\longrightarrow} \Delta$ over Δ. But τ is a limiting cone. So there exists a unique arrow $h : F(x) \longrightarrow a$ making all induced diagrams commute (ie. $h\tau = \sigma^*$).



By adjunction we get a unique arrow $h_* : x \longrightarrow G(a)$. One must check that $(h\tau)_* = h_*(\tau G) = \sigma$.



So h is a unique arrow making all diagrams commute. Hence $\tau G : G(a) \stackrel{\cdot}{\longrightarrow} (\Delta G)$ is a limiting cone, as we wanted to show.

Some other important results about limits

- $Hom(A, \_) : \mathbb{C} \longrightarrow \mathrm{Set}$ preserves all existing limits.
- $\mathrm{Set}^{\mathbb{C}^o}$ is complete.
- A category with a terminal object and pullbacks is complete.
- A category with equalizers and all small products is complete.

(see Borceux)

Monads

We saw that in a Galois correspondence $F \dashv G$ between $\mathbb{A}$ and $\mathbb{B}$ we can conclude that for every A in $\mathbb{A}$

$$A \leq FG(A) \text{ and } FGFG(A) \leq FG(A)$$

We have similar results for adjoints. A straightforward argument from the definition yields natural transformations $\mathbf{1}_{\mathbb{A}} \longrightarrow FG$ (namely the unit) and $FGFG \longrightarrow FG$. These properties are of independent interest.

A Functor $T : \mathbb{A} \longrightarrow \mathbb{B}$ is a monad if it is equipped with natural transformations

- $\eta : \mathbf{1}_{\mathbb{A}} \longrightarrow T$, and
- $\mu : TT \longrightarrow T$

making the following diagrams commute.

$$\begin{array}{ccc} T & \xrightarrow{\eta T} & T^2 \\ {\scriptstyle T\eta}\big\downarrow & \searrow^{\mathbf{1}} & \big\downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

and

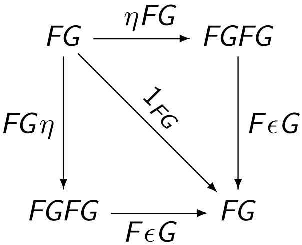
$$\begin{array}{ccc} T^3 & \xrightarrow{\mu T} & T^2 \\ {\scriptstyle T\mu}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

Theorem (Huber)

If $F \dashv G$ then $\langle FG, \eta, F\epsilon G\rangle$ is a monad.

- From $GF \xrightarrow{\epsilon} id$ we get
- $FGF \xrightarrow{\epsilon_{F(\_)}} F$, i.e. $FGF \xrightarrow{F\epsilon} F$. Now apply G
- $FGFG \xrightarrow{F\epsilon G} FG$.

Recall: The adjunction (F, G, η, ϵ) satisfies $(\eta F)(F\epsilon) = 1_F$ and $(G\eta)(\epsilon G) = 1_G$
We must now check the commutativity of



i.e.

- $(\eta FG)(F\epsilon G) = 1_{FG}$, which, by functoriality of G is $(\eta F)(F\epsilon)G$ which is $\mathbf{1}_F G$, i.e. 1_{FG}, and

- $(FG\eta)(F\epsilon G) = 1_{FG}$ which is also immediate: $F(G\eta)(\epsilon G) = F1_G = 1_{FG}$.

For the *join* law, the second commutative square, see [MacLane].

Examples

- The closure operator in a topological space: Let $T(S)$ for any subset S of X be the intersection of all closed sets containing S. $T : \langle P(X), \subseteq\rangle \longrightarrow \langle P(X), \subseteq\rangle$ is a monad: $S \subseteq T(S)$ and $T(T(S)) \subseteq T(S)$

More examples

- Let $\mathcal{M}$ be a monoid with unit 1 and underlying set M. For each set S define $T(A) = M \times A$ and $\eta_A : A \longrightarrow M \times A$ via $\eta_A(a) = (a, 1)$ and $\mu(A) : M \times (M \times A) \longrightarrow M \times A$ via $\mu(m_1, (m_2, a)) \longrightarrow (m_1 m_2, a)$.
  Let $T(A_1 \xrightarrow{\alpha} A_2)$ be the function $(m, a) \mapsto (m, \alpha(a))$. $\langle T, \eta, \mu \rangle$ defines a monad. The monad laws here stipulate $m1 = m = 1m$ and $(m_1 m_2)m_3 = m_1(m_2 m_3)$ (hence the names *unity* and *associative* laws).

More examples

Let P be the covariant power set functor Set $\longrightarrow$ Set mapping sets A to $P(A)$ and maps $A \xrightarrow{f} B$ to the function $P(f) : P(A) \xrightarrow{Im_f} P(B)$ sending each subset of A to its image under f. (This map is sometimes called $\exists_f$). Define the natural transformations $\eta : 1 \longrightarrow P$ and $\mu : PP \longrightarrow P$ by: $\eta_A(x) = \{x\}$ and $\mu S = \bigcup S$

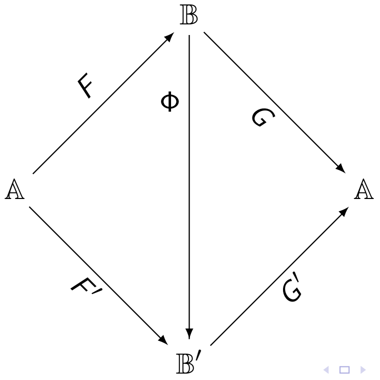
The contavariant Power Set functor

Define $\wp : \text{Set} \longrightarrow \text{Set}$ on objects as before. However, put $\wp(A \xrightarrow{f} B) = \wp(B) \xrightarrow{f^{-1}} \wp(A)$ We will revisit this functor soon.

Does every monad arise as a composition of adjoint functors?

Adjoint resolutions

The answer is yes, given a monad (T, η, μ) on $\mathbb{A}$ there is a *category* of so-called *adjoint resolutions* $(\mathbb{B}, F, G, \epsilon)$ of a monad. These are given by an adjoint pair of functors F, G from $\mathbb{A}$ to $\mathbb{B}$ with unit η and counit ϵ satisfying $F\epsilon G = \mu$. Arrows in this category are morphisms $\Phi : (\mathbb{B}, F, G, \epsilon) \longrightarrow (\mathbb{B}', F', G', \epsilon')$ that satisfy:



The category of adjoint resolutions of T has

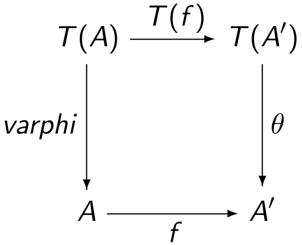
- An initial object $\mathbb{A}_T$ called the *Kleisli* category of T (the category of free T-algebras)
- A final object $\mathbb{A}^T$ called the Eilenberg-Moore category of T (the category of T-algebras).

Given an adjoint resolution $(\mathbb{B}, F, G, \epsilon)$ for T, the unique arrows $K^T : \mathbb{B} \longrightarrow \mathbb{A}^T$ and $K_T : \mathbb{A}_T \longrightarrow \mathbb{B}$ are called the *comparison functors*. The case where the comparison functors give rise to an equivalence or isomorphism of categories is of special interest. (See MacLane or Lambek-Scott for details).

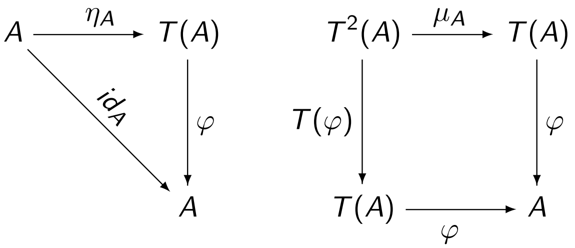
The Eilenberg-Moore category

Given a monad $\langle T, \eta, \mu \rangle$ with $T : \mathbb{A} \longrightarrow \mathbb{A}$ the Eilenberg Moore category for T, denoted $\mathbb{A}^T$ has

- objects: T-algebras, i.e. arrows $T(A) \xrightarrow{\varphi} A$ in $\mathbb{A}$ for some object A satisfying the conditions specified on the next slide.
- Arrows: between two algebras $T(A) \xrightarrow{\varphi} A$ and $T(A') \xrightarrow{\theta} A'$ are $\mathbb{A}$ - arrows $f : A \longrightarrow A'$ making the following square commute:



T-algebras $T(A) \xrightarrow{\varphi} A$ must also satisfy the conditions displayed in the following commutative diagrams



The Kleisli category

Given a monad $\langle T, \eta, \mu \rangle$ on $\mathbb{A}$, the Kleisli category $\mathbb{A}_T$ has as

- *objects*: the objects of $\mathbb{A}$
- arrows from A to A' are arrows α in $\mathbb{A}$ from A to $T(A')$. They compose as follows: $(A \xrightarrow{\alpha} A') \star (A' \xrightarrow{\alpha} A'')$ is the composition (in $\mathbb{A}$)
  $(A \xrightarrow{\alpha} T(A'))(T(A') \xrightarrow{T(\alpha')} T^2(A'') \xrightarrow{\mu_{A''}} T(A'')$

Thus for example, the identity arrow on A in $\mathbb{A}_T$ is $\eta_A : A \longrightarrow T(A)$ (this has to be checked).

⋆Exercises

- What are the algebras of the power set monad P on Set?
- What is its Kelisli category?

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